NOTE TO FILE:
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Dated: 170507 R2:170510 R3: 170518

On Efficiency Formulae, and Growth or Degrowth

Frontispiece

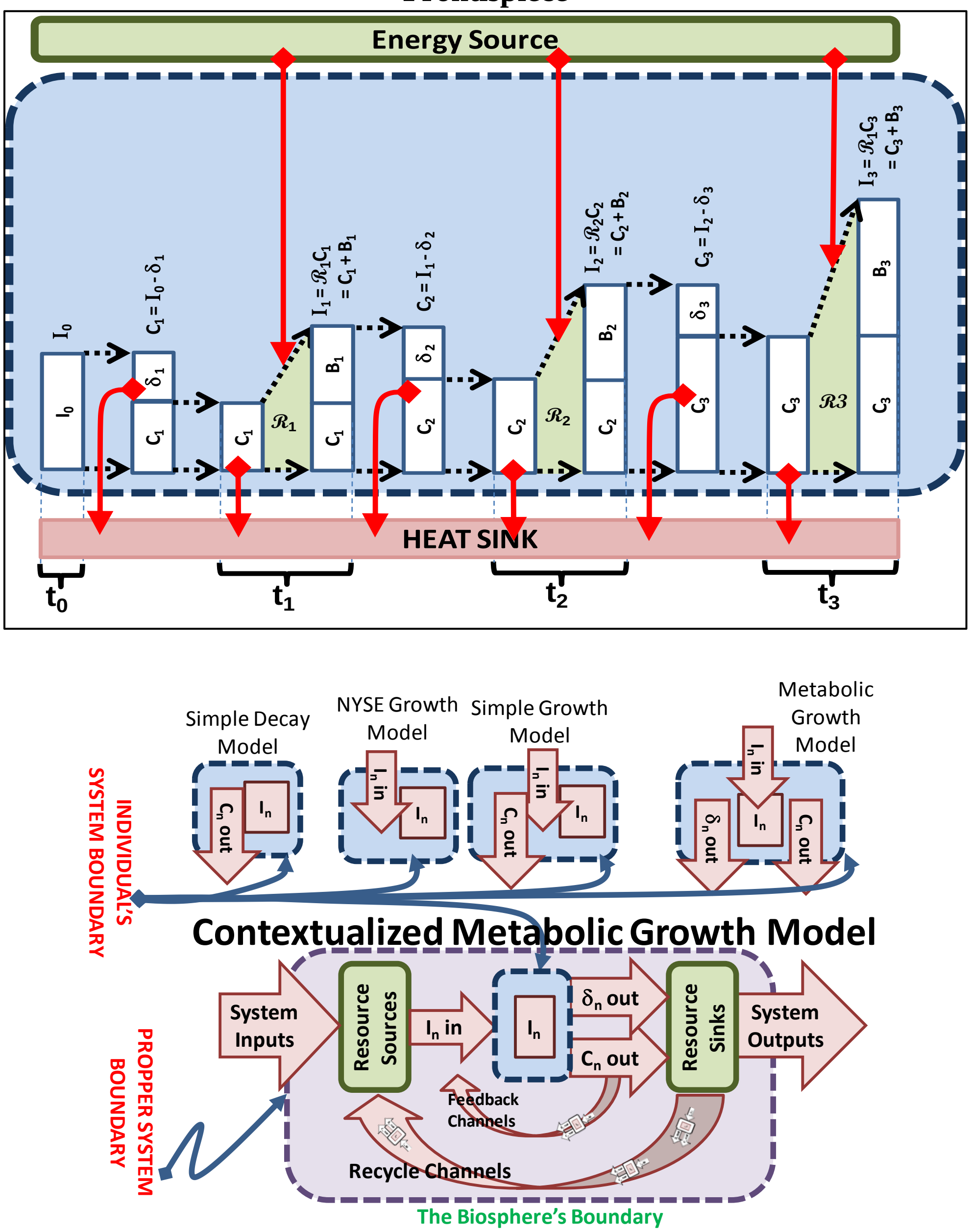

Table of Contents

1 - References

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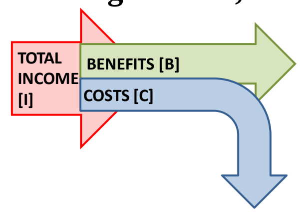
2 - Background

In the Ref C diary note I pursue the question of efficiencies compounded along energy chains (or supply chains). While working on that NTF I decided that I needed to go back and review the relationships between the three relevant formulae for efficiency, and now there are three related notes. In the Ref A diary note I review the definition of I, B and C, as shown in Figure 01, and using those I define the three measures of efficiency η, $\mathcal{R}$ and $\mathcal{N}$. I then develop the conversion formulae, for η, $\mathcal{R}$ and $\mathcal{N}$, as shown in Table 01. Finally, I outlined the two paradigms implicit in the formulae for η and $\mathcal{R}$.

Figure 01 – Diagram of I, B and C



From the Ref A NTF. I = B + C.

Table 01 – Definitions.

Definitions of Flows		
I = Income or Gross Returns	C = Costs or Investments	B = Benefits, Profits, or Net Returns
Definitions of Efficiency Formulae		
$\eta \equiv \frac{B}{I}$	$R \equiv \frac{I}{C}$	$N \equiv \frac{B}{C}$

From the Ref A NTF.

But, in the Ref A NTF I have barely addressed the issue of time. Certainly time comes into the measurement of power, and time passes as energy is consumed, so the passage of time is implicit in all of the discussion there. But I did not go into detail.

2.1 - Two aspects of time

There are two aspects of time are not addressed in that Ref A NTF:

A) I did not address the recurring investments over time, generating recurring income and recurring benefits, by a single agent. This would, for example, be the case for a single organism eating on a daily basis. It would seem that there are two implied instances to look at: maximum possible growth, and maximum possible de-growth. And then, there is everything in between.

B) Nor did I address sequential transformations as energy (or capital) is transferred down a chain from agent to agent. This would, for example, be the case for a trophic chain in which energy is passed from primary producers to apex predators. Or it would describe a supply chain associated with a manufacturing firm.

It is aspect A) of the passage of time (recurring investments over time) that is the topic of this NTF. It is aspect B) of the passage of time that is the topic of the Ref C NTF. In consideration, then, of what I am calling Aspect A, the two competing paradigms for efficiency that were identified in the Ref A NTF both play a role in this aspect of the effects of time. I.e recurring investments can lead to either growth or decay of the initial endowment of resources. Here is the time-related issue that bothers me, as precisely as I can state it:

- When Dr Hall talks about EROI, we start with a small input and end up with a larger output. It would seem that the concept of augmentation of the pre-existing pool of accessible energy is intrinsic to the definition of EROI, herein renamed as Gross ROI, and symbolized as $\mathcal{R}$. He is not, obviously, creating energy out of nothing. He is implicitly capturing or releasing energy from a pre-existing pool or flow of energy. If one wants to take a systems view, one must encompass that pre-existing pool of energy within the boundaries of the system, model the draw against it, and determine the constraints on that draw.
- But, when Dr Odum discussed efficiency, herein symbolized as η, he was talking about the partial consumption of a pool or flow of energy that was already captured and under the control of the organism. It would seem that the concept of diminution of the pre-existing pool of accessible energy is intrinsic to the definition of η, herein called Odum's efficiency. Again, to have a systems view of the process, we need to define a system boundary, and the flows and stores of energy in that context.

Since $\mathcal{N} = \mathcal{R} - 1$ is a simple translation, I can focus on $\mathcal{R}$ and η, thinking that $\mathcal{N}$ comes along for the ride. I therefore don't discuss $\mathcal{N}$ in this NTF.

2.2 - Three NTFs

So, in the Ref A NTF (Part I of a series of three NTFs), I have shown that these two definitions of efficiency, $\mathcal{R}$ and η, are intimately related, and are interchangeable when it comes to analysis of power vs efficiency curves, and yet seem to be intrinsically at odds with one another.

In this NTF (Part II of a series of three) I will examine Aspect A of the time-related dynamics that are implied in the formulation of $\mathcal{R}$ and η.

In the Ref C NTF (Part III of the series) I will focus on Aspect B of the two aspects of time-related dynamics.

3 - Purpose

This NTF is a continuation of the Ref A NTF, and a prelude to the Ref C NTF. The purpose is to explore and clarify the implicit time-related augmentation or diminution of energy pools that exists within the definitions of η and $\mathcal{R}$, and to define the implications for relevant systems and system boundaries.

4 - Discussion

4.1 - Why Do This? Where is it going?

There are three reasons why I think it is important to explore the nature of growth, as formulated in the various formulae for efficiency that I am studying.

First Reason: Back in January Dr Hall asked me to comment on a fascinating draft paper on which he was working with Dr Brown, and that got me re-learning stuff about allometry. Dr Hall's question was:

Garvin
In your maximum power work do you see any reason that power should scale to the ¼ or ¾ power with organism size?
Charlie

His draft paper is at Ref D, my research into the question is at Ref E, and my comments on the draft paper are at Ref F. The empirical data cited in his paper seems to show an obvious biosphere-wide allometric scaling of ¼. I was able to come up with a scaling of 2/3, but not ¼, so I wasn't much help, I suppose. But this did get me thinking about allometric scaling of corporations. That leads to questions about growth, maximum size, and possible constraints on size of corporations, and Galbraith.

Second Reason: In my reading about Galbraith's "theory of the firm" (see Ref A for more details), I came to understand that the mechanisms that NCE theory claims provide constraints on the size of firms do not, in fact, have that effect. Galbraith argues that large firms clearly operate in breach of NCE theory. For example, they minimize the returns to shareholders as if it is a metabolic cost (represented by the δ_i in the following analyses) and maximize the "retained earnings" to use them for persistence (e.g. growth and mergers and acquisitions). This phenomenon is clearly a part of the physiological dynamics of most animals, and now seems to be part of the social dynamics of corporations. In other words, it seems that the reasons that Galbraith cites for the different focus of a modern firm are closely associated with my understanding of the MPP. So Galbraith's arguments (a) show that NCE has a fault (cannot describe limits to corporate growth, nor the shift towards retained earnings), and also (b) show that Odum's MPP has something to say to explain the dynamics that NCE cannot explain. I explored this concept a little when preparing for the Ref G presentation to ISEE 2016, and I explored it a little further when writing the Ref A companion to this NTF.

Third Reason: Finally, there is something seriously wrong with the concept of the "time value of money". In my many years as a project manager I had a focus on discounted cash flows (DCFs). A DCF is essentially just a recursive variation on an ROI. It was considered professionally negligent for a project manager not to consider them. But, they did not work very well, being at best an extremely blunt instrument, and most often a misrepresentation. When I was studying for CATM certification via U Carleton, one professor proposed an improvement on DCFs by using different discount rates for income streams and cost streams, essentially including different risk premiums in each type of cash flow. That made some sense, but it did not address my fundamental discomfort with the idea that "money grows", essentially out of nothing, and that there is a "time value of money". This concept of growing money is, of course, based on the practice of charging interest on loans, but it appears to fly in the face of the 2nd law of thermodynamics. DCFs resolve the problem of the "diamond-water" paradox (see Ref H) by simply ignoring it.

I wonder if a systems approach will help. I am at a stage in the investigation for which I think I can define the system boundary. Once I do that, the results will be comparable to my various economic models. I.e. I wonder if a re-think of decay and/or growth from the ground up, from the perspective of biophysical phenomena with a systems view, might provide some insight into:

- the allometric scaling of modern mega-firms;
- the limits to growth of modern mega-firms; and

- the means to find a replacement for DCFs in financial studies.

4.1.1 - Some Generic Let Statements

Recursive definitions of mathematical processes are not always useful, but sometimes they are. A recursive definition includes an initial value, and a means to calculate the next value from the previous value. Here are some generic let statements for the recursive mathematical definitions that I can put together:

- **Time Markers**:
  - **Initialization** – Let t_0 represent the time marker at the start of the process, and
  - **Recursion** – Let t_1, t_2, … t_n represent a sequence of time markers, for 0 <= i <= n. The duration between markers (e.g. t_2-t_1) will vary, so time proceeds at varying paces. The markers might mark, for example, the times when a fish feeds, and the durations would be periods of digestion and rest between feeding times. For simplicity, all durations are non-zero and are sufficiently long to allow for complete digestion of ingested foods.
- **Gross ROI ratios** – Let $\mathcal{R}_i$ represent the EROI of the feeding event that occurs at time t_i. No feeding event occurs, however for time marker t_0.
- **Costs**:
  - **Investments or Costs** – Let C_i represent the energy invested at time marker t_i. This includes energy invested in the hunt, and energy invested in digestion of the captured food.
  - **Cumulative Investments or Costs** – Let ${}_CC_i$ represent the cumulative amount invested by hunt and capture and digestive processes from time markers t_0 to t_i.
- **Returns**:
  - **Gross Returns** – Let I_i represent the total energy garnered due to the investment of C_i at time marker t_i.
  - **Cumulative Gross Returns** – Let ${}_CI_i$ represent the total energy garnered due to the investment of C_i from time markers t_0 to t_i.
- **Benefits**:
  - **Net Returns** – Let B_i represent the net energy garnered due to the investment at time marker t_i, such that $B_i = I_i – C_i$.
  - Cumulative Net Returns – Let ${}_CB_i$ represent the accumulated benefits accessible from time marker t_0 to t_i.
- **Other Leakage**:
  - **Metabolic Leakage** – Let δ_i represent the energy degraded by metabolic processes during the duration or interval of time $[t_{i-1} – t_i)$
  - **Cumulative Metabolic Leakage** - Let ${}_C\delta_i$ represent the cumulative amount degraded by metabolic processes other than digestion.

As indicated in the Ref A discussion of two paradigms, I think the growth paradigm is the more broad concept, so I will start with the decay paradigm, and nail that one down if I can, before moving on to the more general case.

4.2 - Decay – Declining Values

When using the efficiency ratio η one is in the frame of mind of expecting the value or potency of some pool of resources to decline or be diminished. In such cases of declining value the terminology is usually about depreciation, decay, consumption, or, maybe, dispersion. For example we can talk about decay rates or half-lives of radio-active substances, of depreciation rates of the value of capital equipment, or the efficiency of power tools.

4.2.1 - Decay in Terms of Degraded Energy (η)

The most common interpretation of the concept of efficiency is derived, I think, from the work of Sadi Carnot (1824) and his study of thermal heat engines (see Ref I). Further study of Carnot's ideas was made by Clapeyron (1834, Ref J) and then Clausius (1857, Ref K) – two men out of whose work the modern concept of the second law of thermodynamics emerged. However, the same formulae can be used in at least four contexts:

- The degradation of a portion of a quantity of energy during its transformation;
- Depreciation of the value of capital equipment and assets;
- Decay of the potency of radio-active materials;
- Consumption of non-renewable resources through regular or irregular harvesting.

I will focus on the energy context, without any intended loss of generality.

4.2.1.1 - A Shortage of Simple Examples

It is difficult to think of a single simple example which can be used to explain the concept of efficiency of energy conversion:

- I started with the workings of Carnot's heat engine and found it far from simple. It requires a rather comprehensive understanding of the trade-offs between pressure, temperature and volume, and an understanding of reversibility and irreversibility. So I went looking for a more simple example.
- Next I was working on a description of an electric drill, for which the efficiency would be the energy consumed as useful work divided by the total energy consumed (as in $\eta = B/I$). But then I would need to get into philosophical questions such as "Who defines 'useful'?" and "What do you mean by 'consumed'?" I needed to address electrical energy, mechanical energy, waste heat and the transformations between all of them. The explanation started to sound somewhat artificial, since virtually all of the energy is 'consumed' in one way or another, which begs the question as to "What useful energy is conserved?" And that example does little to avoid the complexities implicit in Carnot's heat engine.
- Then I thought a better explanation would come out of a transformation of energy from one common understanding of 'useful' to another common understanding of 'useful' where the usefulness is still intact, i.e. conserved. Such a condition exists when a litre of gasoline might be used to recharge an electric battery. Gasoline can be 'used' to drive a lawn-mower. Electricity can also be 'used' to drive a lawn-mower. The usefulness of the two types of storage are then directly comparable. However, on thinking of it more carefully, this is also far from simple. We start with a kind of Carnot heat engine (the internal combustion engine) that consumes gasoline (totally) and produces heat and drives an electric dynamo, which generates electricity that courses down a wire, that is then converted to chemical potential energy in the battery. There are at least five different energy transformation involved, each with its own efficiency rating, and each with its own definition of 'usefulness'.
- Setting all of those aside, the best example I can think of is Atwood's Machine (the AM). In the example of the AM, energy stored in an elevated weight is 'used' to elevate another weight, and in the process, some energy is consumed. This requires only a knowledge of Newtonian mechanics, and that is taught in grade 11 physics courses.

- But, if I want to model the iterated activities of decay, I need to pass energy from AM to AM, and return again to the need to imagine an open AM, or OAM. That approach is certainly imaginative, and hardly simple. So, it has its own difficulties. However, I have already worked through those difficulties (Ref L and its Refs) and so am comfortable with them.

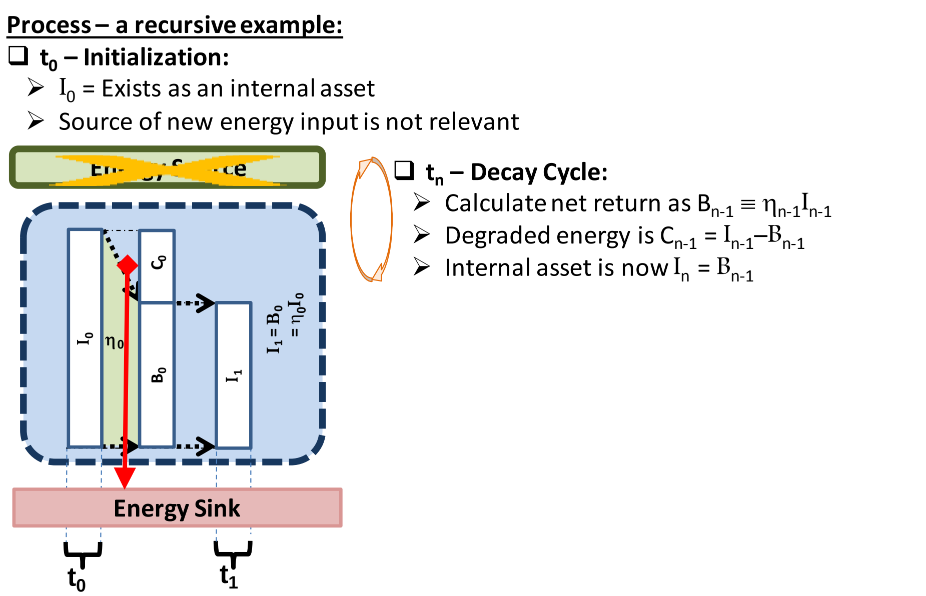
In three years of study, I have travelled full circle and come back to the AM and OAMs. In development of the Ref L application I modeled a quantity of Sunshine being absorbed by primary producers (plants) and then being passed down a chain of consumers. The ideas developed there are the template from which the very simple diagram in Figure 02 is drawn.

4.2.1.2 - The Energy Transformation Example

My approach will be to describe a recursive process that starts with three value I_0, C_0 and B_0, and on each iteration n, will produce a new set of three values: I_n, C_n and B_n. Then I will produce a set of discrete-time formula by which the three values for any n can be computed directly, without the iterative process.

The process of recursion requires a set of initial values, and a set of transition rules that produce the next set of values from the previous values.

Figure 02 – Decay Paradigm – Exploded View



For the recursion process shown in Figure 02 I start with the initial endowment of energy I_0, being the net return of some unspecified process that established the initial endowment. The area in blue represents the system internals, and the dotted line is the system boundary. The external energy source is X-ed out (in orange), because the system is isolated from energy inputs throughout the decay process. However, as energy is degraded, the waste heat produced is delivered into the external energy sink, as indicated by the red arrow. The "still useful" energy is calculated as $\eta_0 I_0$ and denoted by B_0, and the energy that was degraded and sent to the sink is calculated as I_0–B_0 and denoted by C_0. Then, I need a bridging equation for the third value in the form of I_1=B_0. Finally the value of n is advanced from 0 to 1, and the set of transition rules is applied again in a never-ending cycle. The size of the energy pool under control of the agent is I_n at discrete time marker t_n.

The first three iterations are shown in Table 02, and the discrete-time formulae derived from them are in Table 03.

Table 02 - Formulae and Values for a Recursive Regime – Decay Paradigm.	
Initialization Values – Time Marker = t_0:	
Values	Explanations
$0 < I_0 < \infty$	Initial endowment of resources.
$0 < \eta_0 < 1$	Pre-set value, along with all other η_i.
Recursion Input Values and Formulae – Time Marker = t_i:	
Values	Explanations
$1 < \eta_i < \infty$	A series of values indicating the expected efficiency during each iteration would be supplied. This might be constant for all iterations, or vary with time, say, as declining or increasing efficiency. In a financial application, the depreciation rate would be constant over all iterations. When applied to radioactive materials, the decay rate would be constant. But if you are modeling a machine that wears out or clogs up with usage, the values would vary.
Formulae	Explanations
$B_{i-1} = \eta_{i-1} I_{i-1}$	Calculate the portion preserved.
$C_{i-1} = (I_{i-1} - B_{i-1})$	Calculate the portion degraded.
$I_i = B_{i-1}$	Bridge to next iteration.
$i = i + 1$	Advance to next iteration.
Three Iterations – Time Marker = t_i:	
Formulae	Explanations
$B_0 = \eta_0 I_0$	Calculate the portion preserved.
$C_0 = (I_0 - B_0)$	Calculate the portion degraded.
$I_1 = B_0$	Bridge to next iteration.
Formulae	Explanations
$B_1 = \eta_1 I_1$	$B_1 = \eta_1(\eta_0 I_0)$
$C_1 = (I_1 - B_1)$	$C_1 = \eta_0 I_0 - \eta_1(\eta_0 I_0)$
$I_2 = B_1$	$I_2 = \eta_1(\eta_0 I_0)$
Formulae	Explanations
$B_2 = \eta_2 I_2$	$B_2 = \eta_2(\eta_1(\eta_0 I_0))$
$C_2 = (I_2 - B_2)$	C2 = $\eta_1(\eta_0 I_0) - \eta_2(\eta_1(\eta_0 I_0))$
$I_3 = B_2$	I3 = $\eta_2(\eta_1(\eta_0 I_0))$

Table 03 – Discrete-Time Formulae – Time Marker = t_n – Decay Paradigm.		
Formulae	Explanations	
$B_n \equiv I_0 \times \prod_{i=0}^{n} \eta_i$	The net returns at each iteration suffer an exponential decline.	Equ 01
$C_n \equiv I_0(1-\eta_n)\prod_{i=0}^{n-1} \eta_i$	The energy degraded also suffers an exponential decline.	Equ 02
$I_n \equiv I_0 \times \prod_{i=0}^{n-1} \eta_i$	And, the energy available for use has an exponential decline. I note that $I_n = B_n + C_n$.	Equ 03

These results are as expected. When considering cumulative values, the only item of interest would be the cumulative value of C (denote it as ${}_CC_n$) as n approaches infinity. Since there is no energy going into the system, the total that has left the system is the only sum that might be of interest.

Table 04 – Cumulative Discrete-Time Formulae – ${}_CC_n$ – Decay Paradigm.		
$C_0 = (I_0 - B_0)$	$C_0 = I_0 - \eta_0 I_0$	
$C_1 = (I_1 - B_1)$	$C_1 = \eta_0 I_0 - \eta_1(\eta_0 I_0)$	
$C_2 = (I_2 - B_2)$	$C_2 = \eta_1(\eta_0 I_0) - \eta_2(\eta_1(\eta_0 I_0))$	
$C_3 = (I_3 - B_3)$	$C_3 = \eta_2(\eta_1(\eta_0 I_0)) - \eta_3(\eta_2(\eta_1(\eta_0 I_0)))$	
${}_CC_3 = C_0+C_1+C_2+C_3$	${}_CC_3 = I_0 - \eta_0 I_0$ $+ \eta_0 I_0 - \eta_1(\eta_0 I_0)$ $+ \eta_1(\eta_0 I_0) - \eta_2(\eta_1(\eta_0 I_0))$ $+ \eta_2(\eta_1(\eta_0 I_0)) - \eta_3(\eta_2(\eta_1(\eta_0 I_0)))$	
${}_CC_n = I_0\left[1+\sum_{i=0}^{n-1}\left(\prod_{j=0}^{i}(\eta_j)\right)-\sum_{i=0}^{n}\left(\prod_{j=0}^{i}(\eta_j)\right)\right]$		Equ 04
${}_CC_n = I_0\left[1-\prod_{i=0}^{n}(\eta_i)\right]$		Equ 05
$\lim_{n\to\infty} {}_CC_n = I_0 \quad IF \quad \lim_{n\to\infty} \eta_i \neq 1$		Equ 06

Equation 04 simplifies to Equation 05. As i approaches infinity, the product approaches zero, and ${}_CC_n$ approaches I_0. This makes sense. The limit of ${}_CC_n = I_0$ in all cases except where the limit of η_n exists and is equal to 1.

4.3 - Simple Growth – Augmentation of Capture Rates

When using the efficiency ratio $\mathcal{R}$ one is in the frame of mind of expecting a positive return on an investment. In other words, the expected outcome is growth of the pool of resources. When considering growth I then naturally turn to the ratio $\mathcal{R}$, which usually goes by the names of EROI, or EROEI, or ROI, or the gross return on investment

4.3.1 - Simple Growth in terms of Gross Returns ($\mathcal{R}$)

As for the decay paradigm, the growth paradigm can be used in many contexts such as:

- The growth of a cell after mitosis;
- The growth of an organism after birth;
- The growth of a corporation, city, or society; or
- The growth of an investment portfolio.

I will continue the discussion here using the context of energy consumption, choosing the context of the growth of an organism that feeds irregularly, using the terminology associated with EROI – i.e. energy returned on energy invested.

When considering the use of the efficiency ratio $\mathcal{R}$, one must invest some energy that is already under the control of the investor, in the hopes of later garnering a share of the as-yet-unexploited resource and bringing it under control. So, consider a large pre-existing source (pool or flow) of energy that a species has not yet tapped, and is not yet under the control of the species. An example would be a flow down a stream of edible insect larvae not yet eaten by trout.

$\mathcal{R}$ is then a subjective measure of growth, from the perspective of the "investor". One must garner at least the amount invested or suffer a loss. If one manages to garner more than the amount invested, then $\mathcal{R}$ will be greater than 1. Otherwise, $\mathcal{R}$ will be less than 1. In this analysis, $\mathcal{R}$ is always treated as greater than 1, but without loss of generality. I.e. I believe the logic is the same when $\mathcal{R}$ is between 0 and 1.

The presentation in this section will be very similar to the presentation in section 4.2.

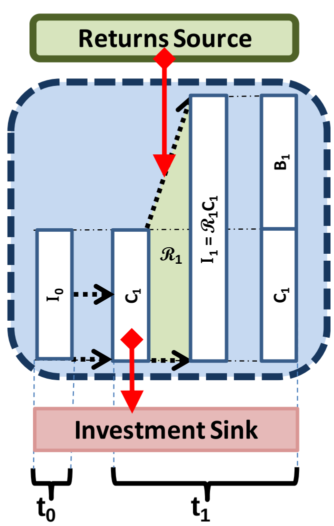
The process of recursion requires a set of initial values, and a set of transition rules that produce the next set of values from the previous values.

For the recursion process shown in Figure 03 I start with the initial endowment of energy I_0, being the net return of some unspecified process that established the initial endowment. The area in blue represents the system internals,

Figure 03 – Simple Growth Paradigm – Exploded View

Process – a recursive example:

- ❑ **t_0 – Initialization:**
  - ➢ Source exists as store of external assets
  - ➢ I_0 = Exists as internal asset
- ❑ **t_n – Growth Cycle:**
  - ➢ Relabel I_{n-1} as investment "cost" C_n
  - ➢ Expend as C_n as "investment"
  - ➢ Calculate gross returns as $I_n \equiv \mathcal{R}_n C_n$
  - ➢ Calculate net return as $B_n \equiv I_n - C_n$



and the dotted line is the system boundary. The external energy source is active, because the system is open to energy inputs throughout the growth process, as indicated by the upper red arrow. As energy is "invested", the waste heat produced is delivered into the external energy sink, as indicated by the lower red arrow. The "still useful" energy is calculated as $\mathcal{R}_1C_1$ and denoted by I_1, and the energy that was degraded and sent to the sink is the entirety of the resource available for investment, denoted by C_1. In the decay paradigm my bridging equation comes last, but in the simple growth paradigm is comes first, where I_{n-1} is relabeled as C_n. Finally the value of n is advanced from 0 to 1, and the set of transition rules is applied again in a never-ending cycle. The size of the energy pool under control of the agent is I_n at discrete time marker t_n.

The first three iterations are shown in Table 05, and the discrete-time formulae derived from them are in Table 06.

Table 05 - Formulae and Values for a Recursive Regime – Simple Growth Paradigm.	
Initialization Values – Time Marker = t_0:	
Values	Explanations
$0 < I_0 < \infty$	Initial endowment of resources.
$0 < R_0 < 1$	Pre-set value, along with all other $\mathcal{R}_i$.
Recursion Input Values and Formulae – Time Marker = t_i:	
Values	Explanations
$1 < R_i < \infty$	A series of values indicating the expected EROI during each iteration would be supplied. This might be constant for all iterations, or vary with time, say, as declining or increasing EROI. In a financial investment such as a long-term bond, the ROI might be constant over all iterations. But if you are modeling extraction of a non-renewable resource with a "best first" approach, the EROI will vary with time.
Formulae	Explanations
$C_i = I_{i-1}$	Bridge from the previous iteration.
$I_i = R_iC_i$	Calculate the gross returns on the investment.
$B_i = I_i - C_i$	Calculate the net returns on the investment.
$i = i + 1$	Advance to next iteration.
Three Iterations – Time Marker = t_i:	
Formulae	Explanations
$C_1 = I_0$	$C_1 = I_0$
$I_1 = R_1C_1$	$I_1 = \mathcal{R}_1I_0$
$B_1 = I_1 - C_1$	$B_1 = \mathcal{R}_1I_0 – I_0$
Formulae	Explanations
$C_2 = I_1$	$C_2 = \mathcal{R}_1I_0$
$I_2 = R_2C_2$	$I_2 = \mathcal{R}_2\mathcal{R}_1I_0$
$B_2 = I_2 - C_2$	$B_2 = \mathcal{R}_2\mathcal{R}_1I_0 – \mathcal{R}_1I_0$
Formulae	Explanations
$C_3 = I_2$	$C_3 = \mathcal{R}_2\mathcal{R}_1I_0$
$I_3 = R_3C_3$	$I_3 = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1I_0$
$B_3 = I_3 - C_3$	$B_3 = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1I_0 – \mathcal{R}_2\mathcal{R}_1I_0$

Using Table 05 I can construct discrete-time formulae for each of C, I and B, as shown in Table 06.

Table 06 – Discrete-Time Formulae – Simple Growth Paradigm.		
Formulae	Explanations	
$C_n \equiv I_0 \prod_{i=1}^{n-1} R_i$	The costs undergo an exponential increase.	Equ 07
$I_n \equiv I_0 \prod_{i=1}^{n} R_i$	The gross returns undergo an exponential increase.	Equ 08
$B_n \equiv I_0 (R_n - 1) \prod_{i=1}^{n-1} R_i$	And, the net returns undergo an exponential increase. I note that $I_n = B_n + C_n$.	Equ 09

Again, these results are as expected. Since the entirety of the assets currently under control (C_n) is invested, this represents some estimate of the maximal growth possible. Any competition with other organisms for resources would be reflected in depressed values for $\mathcal{R}_n$, so such depressed growth dynamics could be suitably modeled. When considering cumulative values, all three associated cumulative values will rise exponentially.

Table 07 – Cumulative Discrete-Time Formulae –Simple Growth Paradigm.		
$_CC_n$ – Cumulative Costs.		
$C_1 = I_0$	$C_1 = I_0$	
$C_2 = I_1$	$C_2 = \mathcal{R}_1 I_0$	
$C_3 = I_2$	$C_3 = \mathcal{R}_2 \mathcal{R}_1 I_0$	
$_CC_3 = C_1 + C_2 + C_3$	$_CC_3 = I_0$ $+ \mathcal{R}_1 I_0$ $+ \mathcal{R}_2(\mathcal{R}_1 I_0)$	
$_CC_n = I_0 \left[1 + \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} (R_j) \right) \right]$		Equ 10
$\lim_{n\to\infty} {}_CC_n = \infty \quad IF \quad \lim_{n\to\infty} R_i \neq 1$		Equ 11
$_CI_n$ – Cumulative Gross Returns.		
$I_1 = R_1 C_1$	$I_1 = \mathcal{R}_1 I_0$	
$I_2 = R_2 C_2$	$I_2 = \mathcal{R}_2 \mathcal{R}_1 I_0$	
$I_3 = R_3 C_3$	$I_3 = \mathcal{R}_3 \mathcal{R}_2 \mathcal{R}_1 I_0$	
$_CI_3 = I_1 + I_2 + I_3$	$_CI_3 = \mathcal{R}_1 I_0$ $+ \mathcal{R}_2(\mathcal{R}_1 I_0)$ $+ \mathcal{R}_3(\mathcal{R}_2(\mathcal{R}_1 I_0))$	
$_CI_n = I_0 \sum_{i=1}^{n} \left(\prod_{j=1}^{i} (R_j) \right)$		Equ 12
$\lim_{n\to\infty} {}_CI_n = \infty \quad IF \quad \lim_{n\to\infty} R_i \neq 1$		Equ 13

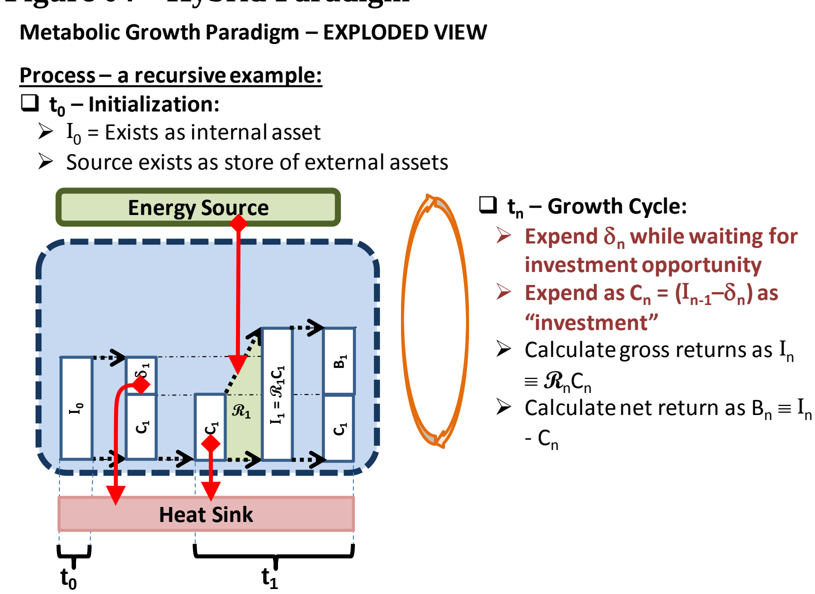
$_CB_n$ – Cumulative Net Returns.	
$B_1 = I_1 - C_1$	$B_1 = \mathcal{R}_1 I_0 - I_0$
$B_2 = I_2 - C_2$	$B_2 = \mathcal{R}_2\mathcal{R}_1 I_0 - \mathcal{R}_1 I_0$
$B_3 = I_3 - C_3$	$B_3 = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1 I_0 - \mathcal{R}_2\mathcal{R}_1 I_0$
$_CB_3 = B_1+B_2+B_3$	$_CB_3 = \mathcal{R}_1 I_0 - I_0$ $+ \mathcal{R}_2\mathcal{R}_1 I_0 - \mathcal{R}_1 I_0$ $+ \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1 I_0 - \mathcal{R}_2\mathcal{R}_1 I_0$
$_CB_n = I_0\left[\sum_{i=1}^{n}\left(\prod_{j=1}^{i}(R_j)\right) - \sum_{i=1}^{n-1}\left(\prod_{j=1}^{i}(R_j)\right) - 1\right]$	Equ 14
$_CB_n = I_0\left[\prod_{i=1}^{n}(R_i) - 1\right]$	Equ 15
$\lim_{n\to\infty} {_CB_n} = \infty \quad IF \quad \lim_{n\to\infty} R_i \neq 1$	Equ 16

Equation 14 simplifies to Equation 15 since many of the terms in the precursor simply cancel each other out. Compare Equations 14 and 15 with Equations 04 and 05. WRT Equ 16, in the limit, as n approaches infinity, the product approaches infinity, as long as $\mathcal{R}_i$ remains greater than 1. This makes sense.

4.4 - Growth With Metabolic Costs – A Hybrid Paradigm

On Page 55 of the Ref M text (Otto and Day, 2007) the authors describe a wide range of "classic" models in ecology and evolutionary biology. I believe the models considered in sections 4.2 and 4.3 above fall into the categories of exponential growth or possibly logistic growth. When I add metabolic leakage to the dynamics, we move into the category of consumer-resource equations. It would seem from a reading of the material in that Ref M text that this territory has been thoroughly trodden before, as these are now considered "classic" models. But I am unsure whether the explicit use of gross returns ratio $\mathcal{R}$ was considered in those presentations. Certainly, I find no mention of either gross returns or net returns in the glossary of the book, nor in the discussion of these classic models following page 55. Furthermore, the presentations of those models in the text are all continuous-

Figure 04 – Hybrid Paradigm



time models, and the growth rates seem to be constants in the model. I want to consider a varying EROI (e.g. best and easiest first) in a discrete-time model. So, it seems, I need to wander away from the classic models as presented in the Ref M book.

Figure 04 shows an exploded view of the recursive regime. Time proceeds in asynchronous discrete steps along the bottom, with periods of inactivity between time markers, during which some waste heat is produced via metabolic maintenance. The remaining energy C_n is invested in hunting and digestion (flowing to the heat sink), and the gross returns (I_n) are used to replace C_n and augment it by an amount B_n. It is not lost on me that I could include η_1 as the efficiency factor between I_0 and C_1, but it complicates the diagram and is not entirely necessary. I decided to take δ_n as input in place of η_n. I may need to revisit that decision later.

"Growth with metabolic costs" is meant to be a generic phrase that includes related dynamics such as "growth with harvesting". In either case, you have growth phases inter-leaved with phases in which resources are removed out of the control of the investor. The "metabolic costs" can be considered analogous to friction. They are unavoidable costs that evolutionary pressures tend to minimize as the struggle for persistence continues.

Table 08 - Formulae and Values for a Recursive Regime – Growth With Costs Paradigm.		
Initialization Values – Time Marker = t_0:		
Values	Explanations	
$0 < I_0 < \infty$	Initial endowment of resources.	Equ 17
$C_0 = B_0 = 0$	Set to zero.	Equ 18
$R_0 and \delta_0$	Not meaningful – not set.	Equ 19
Recursion Values and Formulae – Time Marker = t_i:		
Values	Explanations	
$1 < R_i < \infty$	Expected gross return on investment. The principle of "best and easiest first" might mean this declines with time. But my focus on growth in this section means it is always greater than one.	Equ 20
$0 < \delta_i < I_{i-1}$	This reduces the amount available for investment. It could be modeled as a fixed or variable fraction of I_{i-1}. I have chosen to be more general and identify it as an input with range restricted by the size of I_{i-1}.	Equ 21
Formulae	Explanations	
$C_i = (I_{i-1} - \delta_i)$	The energy available for investment is the remaining pool from the previous hunt, less the metabolic costs of resting between hunts.	Equ 22
$I_i = (R_i \times C_i)$	The current pool is established as the gross returns on the investment C_i.	Equ 23
$B_i = (I_i - C_i)$	Net returns on the investment C_i.	Equ 24

Table 08 - Continued		
Cumulative Formulae – Basic Definitions – Time Marker = t_n:		
Formulae	Explanations	
$_{C}C_n \equiv \sum_{i=1}^{n} C_i$	The energy available for investment is the remaining pool from the previous hunt, less the metabolic costs of resting between hunts.	Equ 25
$_{C}I_n \equiv \sum_{i=1}^{n} I_i$	Gross returns on the investment C_i.	Equ 26
$_{C}B_n \equiv \sum_{i=1}^{n} B_i$	Net returns on the investment C_i.	Equ 27

Figure 05 – Three Iterations of Recursion Process

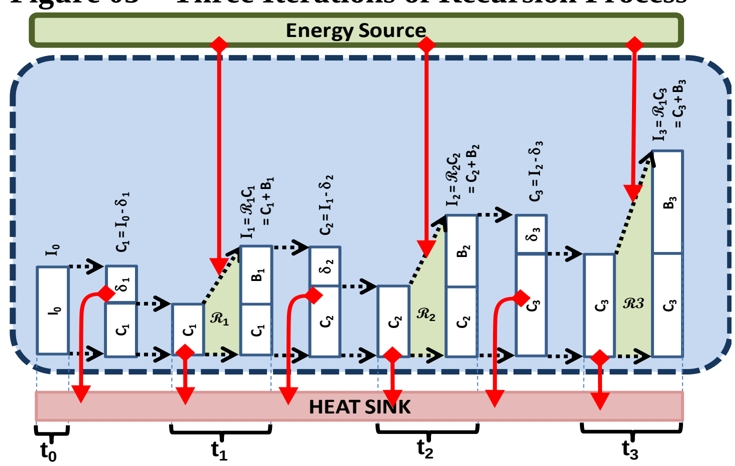


Figure 05 shows an exploded view of three iterations of the recursion process for the hybrid "Growth With Metabolic Costs" paradigm.

I cannot use this recursive regime to produce analytic formulae that are continuous in time, since the duration of time between time markers is assumed to vary, with the intention of keeping the model as general as possible. Also, the $\mathcal{R}_i$ and δ_i series of inputs are not specified as functions of time, again with the intent of keeping this as general as possible. However, I should be able to produce some non-recursive discrete-time equations. In Table 06 I did this first with a simple growth process in which there is no degradation of energy for any reasons other than those encoded in the definition of $\mathcal{R}$. I.e. $\delta_i = 0$ for all i, and so there is no leakage of energy from the I_i pool of energy due to metabolic needs between feeding events. I'll do it again now but allowing for leakage. I.e. $0 < \delta_i < I_i$. Compare Equ 28 with Equ 08.

Table 09 – Growth – I_n Expressed Using $\mathcal{R}$. With metabolic leakage.			
Time Marker	**Energy Resulting From Feeding Event**	**Symbolic Expression in $\mathcal{R}$**	
t_0	e_0 (no feeding happens at this time marker)	I_0	
t_1	$C_1 = I_0 - \delta_1$; $I_1 = (\mathcal{R}_1 \times C_1)$	$I_1 = \mathcal{R}_1(I_0-\delta_1)$	
t_2	$C_2 = I_1 - \delta_2$; $I_2 = (\mathcal{R}_2 \times C_2)$	$I_2 = \mathcal{R}_2(\mathcal{R}_1(I_0-\delta_1)-\delta_2)$	
t_3	$C_3 = I_2 - \delta_3$; $I_3 = (\mathcal{R}_3 \times C_3)$	$I_3 = \mathcal{R}_3(\mathcal{R}_2(\mathcal{R}_1(I_0-\delta_1)-\delta_2)-\delta_3)$	
t_n	$C_n = I_{n-1} - \delta_n$; $I_n = (\mathcal{R}_n \times C_n)$ $I_n = \left(I_0 \prod_{i=1}^{n}(R_i)\right) - \sum_{i=1}^{n}\left(\delta_i \prod_{j=i}^{n}(R_j)\right)$		Equ 28

That defines the growth curve for a given series of values for each of $\mathcal{R}_i$ and δ_i. The first term is the same as Equation 08, but the second term is the amount burnt off by metabolic needs, other than digestion.

One way to simplify Equation 28 would be to assume that all of the $\mathcal{R}_i$ are the same, and all of the δ_i are the same. Then Equation 28 would become:

$$I_n = I_0 R^n - \delta \sum_{i=1}^{n} \left(R^i\right)$$	Equ 29

I may be able to produce a discrete-time non-recursive formula for each of the cumulative flows:

Table 10 – Growth – Cumulative Values Expressed Using $\mathcal{R}$. With metabolic leakage.			
Time Mark	**Recursive Energy Formulae Resulting From Feeding Event**	**Symbolic Expression in $\mathcal{R}$**	
t_0	$I_0 > 0$	I_0	
	$C_0 = 0$	0	
	$B_0 = I_0 – C_0 = I_0$	I_0	
	$_C C_0 = 0$	0	
	$_C I_0 = I_0$	I_0	
	$_C B_0 = {_C I_0} – {_C C_0} = I_0$	I_0	
t_1	$C_1 = (I_0–\delta_1)$	$C_1 = (I_0–\delta_1)$	
	$I_1 = \mathcal{R}_1 C_1$	$I_1 = \mathcal{R}_1(I_0–\delta_1)$	
	$B_1 = I_1–C_1 = (\mathcal{R}_1\text{-}1)C_1$	$B_1 = (\mathcal{R}_1–1)(I_0–\delta_1)$	
	$_C C_1 = {_C C_0}+C_1 = C_1$	$_C C_1 = (I_0–\delta_1)$	
	$_C I_1 = {_C I_0}+I_1$	$_C I_1 = I_0+\mathcal{R}_1(I_0–\delta_1)$	
	$_C B_1 = {_C B_0}+B_1$	$_C B_1 = I_0+(\mathcal{R}_1–1)(I_0–\delta_1)$	
t_2	$C_2 = (I_1–\delta_2)$	$C_2 =\mathcal{R}_1(I_0–\delta_1)–\delta_2$	
	$I_2 = \mathcal{R}_2 C_2$	$I_2 = \mathcal{R}_2(\mathcal{R}_1(I_0–\delta_1)–\delta_2)$	
	$B_2 = I_2–C_2 = (\mathcal{R}_2\text{-}1)C_2$	$B_2 = (\mathcal{R}_2–1)(\mathcal{R}_1(I_0–\delta_1)–\delta_2)$	
	$_C C_2 = {_C C_1}+C_2$	$_C C_2 = (I_0–\delta_1)+(\mathcal{R}_1(I_0–\delta_1)–\delta_2)$	
	$_C I_2 = {_C I_1}+ I_2$	$_C I_2 = I_0+\mathcal{R}_1(I_0–\delta_1)+\mathcal{R}_2(\mathcal{R}_1(I_0–\delta_1)–\delta_2)$	
	$_C B_2 = {_C B_1}+B_2$	$_C B_2 = I_0+(\mathcal{R}_1–1)(I_0–\delta_1)+(\mathcal{R}_2–1)+(\mathcal{R}_2–1)(\mathcal{R}_1(I_0–\delta_1)–\delta_2)$	
t_3	$C_3 = (I_2–\delta_3)$	$C_3 = \mathcal{R}_2(\mathcal{R}_1(I_0–\delta_1)–\delta_2)–\delta_3$	Equ 30
	$I_3 = \mathcal{R}_3 C_3$	$I_3 = \mathcal{R}_3(\mathcal{R}_2(\mathcal{R}_1(I_0–\delta_1)–\delta_2)–\delta_3)$	Equ 31
	$B_3 = I_3–C_3 = (\mathcal{R}_3\text{-}1)C_3$	$B_3 = (\mathcal{R}_3–1)(\mathcal{R}_2(\mathcal{R}_1(I_0–\delta_1)–\delta_2)–\delta_3)$	Equ 32
	$_C C_3 = {_C C_2}+C_3$	$_C C_3 = (I_0–\delta_1)+(\mathcal{R}_1(I_0–\delta_1)–\delta_2)$ $+ \mathcal{R}_2(\mathcal{R}_1(I_0–\delta_1)–\delta_2)–\delta_3$	Equ 33

	$_C I_3 = {_C I_2} + I_3$	$_C I_3 = I_0 + \mathcal{R}_1(I_0-\delta_1) + \mathcal{R}_2(\mathcal{R}_1(I_0-\delta_1)-\delta_2)$ $+\mathcal{R}_3(\mathcal{R}_2(\mathcal{R}_1(I_0-\delta_1)-\delta_2)-\delta_3)$	Equ 34
	$_C B_3 = {_C B_2} + B_3$	$_C B_3 = I_0 + (\mathcal{R}_1-1)(I_0-\delta_1) + (\mathcal{R}_2-1)$ $+(\mathcal{R}_2-1)(\mathcal{R}_1(I_0-\delta_1)-\delta_2)$ $+(\mathcal{R}_3-1)(\mathcal{R}_2(\mathcal{R}_1(I_0-\delta_1)-\delta_2)-\delta_3)$	Equ 35

I have seen Equ 31 before, being the precursor to Equation 28. To produce equation 28 I expanded the precursor, grouped terms by sets of factors (products of $\mathcal{R}_i$) and used Σ and Π notation to make it compact and generalized for any value of n. I proceed to do the same here, producing a discrete-time equation for each of equations 30 through 35, but skipping over 31.

$C_n = \left(I_0 \prod_{i=1}^{n-1}(R_i)\right) - \delta_n - \sum_{i=1}^{n-1}\left(\delta_i \prod_{j=i}^{n-1}(R_j)\right)$	Equ 36 From Equ 30

$B_n = (R_n - 1)\left(\left(I_0 \prod_{i=1}^{n-1}(R_i)\right) - \delta_n - \sum_{i=1}^{n-1}\left(\delta_i \prod_{j=i}^{n-1}(R_j)\right)\right)$	Equ 37 From Equ 32

${_C C_n} = I_0\left(1 + \sum_{i=1}^{n-1}\left(\prod_{j=1}^{i}(R_j)\right)\right) - \delta_n - \sum_{i=1}^{n-1}\left(\delta_i\left(1 + \sum_{j=i}^{n-1}\left(\prod_{k=i}^{j} R_k\right)\right)\right)$	Equ 38 From Equ 33

${_C I_n} = I_0 \sum_{i=1}^{n}\left(\prod_{j=1}^{i}(R_j)\right) - \sum_{i=1}^{n}\left(\delta_i \sum_{j=i}^{n}\left(\prod_{k=i}^{j} R_k\right)\right)$	Equ 39 From Equ 34

${_C B_n} = \sum_{k=1}^{n}\left[\left(I_0 \prod_{i=1}^{k-1}(R_i)\right) - \delta_k - \sum_{i=1}^{k-1}\left(\delta_i \prod_{j=i}^{k-1}(R_j)\right)\right]$	Equ 40 From Equ 35

4.4.1 - Growth and Odum's Efficiency (η)

In Table 05 of the Ref A NTF is shown the conversion formula for $\mathcal{R}$ and η as $\mathcal{R} = 1/(1-\eta)$. It is relatively simple to substitute this conversion into all of the above formulae, and they will still be as valid as before. But in normal practice, η is used when the accessible pool of energy is being consumed, and de-growth is the observable pattern. So, I need to think about the meaning of such equations in which η occurs and growth is implied.

For example, equations 13 and 21 are so transformed into equations 26 and 27:

$$I_n = \left(I_0 \prod_{i=1}^{n} \left(\frac{1}{1-\eta_i}\right)\right) - \sum_{i=1}^{n} \left(\delta_i \prod_{j=i}^{n} \left(\frac{1}{1-\eta_j}\right)\right)$$	Equ 41 From Equ 28

$$C_n = \left(I_0 \prod_{i=1}^{n-1} \left(\frac{1}{1-\eta_i}\right)\right) - \delta_n - \sum_{i=1}^{n-1} \left(\delta_i \prod_{j=i}^{n-1} \left(\frac{1}{1-\eta_j}\right)\right)$$	Equ 42 From Equ 36

Such substitutions could be done for all of the equations in the hybrid paradigm. There is a potential source of confusion here that I need to note and keep in mind. There are two sorts of η implied in this hybrid paradigm:

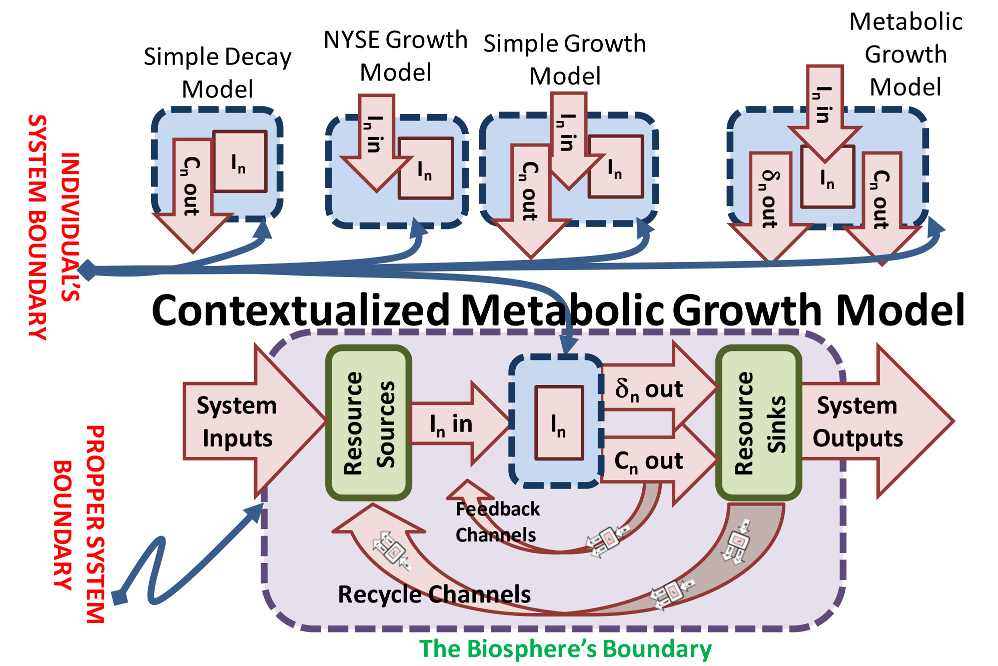
- The efficiency (η_n) that is implicitly associated with each δ_n. When metabolic leakage occurs (i.e. when δ_n energy is lost to the sink) the existing resource pool is diminished, and there is an implicit efficiency associated with that loss. I chose not to encode these as efficiencies in the above equations to preserve the distinction with the next type of efficiency.
- The efficiency ($1/(1-\mathcal{R}_n)$) that is implicitly associated with the gross returns ratio $\mathcal{R}_n$.

4.4.2 - A systems Boundary Diagram

When considering the usual inputs and outputs of a "discounted cash flow" (DCF) calculation, there is only one sort of input (usually dollars) and one sort of output (dollars again) and the inflows are lumped together as income, and the outflows are lumped together as costs. But in a biophysical system there are several large classes of both input and output – differing types of matter sources, sinks and cycles, and differing means of acquiring, burning and disposing of energy.

In all of the above discussion I have concentrated on energy inputs and energy outputs (or cash inputs and cash outputs), but I have taken one step towards this added complexity by including a new output channel. The difference between the DCF model and my metabolic growth model is the added output channel for expenditures related to non-discretionary metabolic maintenance. I have distinguished between output used for discretionary investments, and output caused by the demands of metabolic maintenance. DCF and the simple growth model does not allow for that distinction, as all costs are lumped together as costs. So we get several possible "system diagrams" shown at the top of Figure 05. I have not discussed the NYSE growth model to this point, but I include the system boundary diagram there. The system boundaries are the same, but the interaction of these flows inside or outside of the system boundaries are not visible in the diagram.

Figure 05 – System Boundaries



In other words, perhaps two problems with DCF are the lack of wider context outside of the scope of control, and the lack of attention to the roles played by different flows inside of their accepted scope of control. I think a better perspective is possible if an outer system boundary is also drawn which includes the impact on the sources and sinks, and the feedback from those impacts. I believe that the lower diagram is a generic systems diagram in which the need to examine those impacts becomes more clear. Growth cannot happen independently of the availability of the resources that make it possible, nor of the costs of maintaining increased size. Neither DCF nor any of the other individualized system boundaries are very sensitive to either of these concerns.

5 - Comments and Ideas for Further Action

- I have done my best to get these discrete-time formulae right, and to validate them using MS Excel, but they are difficult to implement in Excel due to interactions of the Σ and Π notations. There is a function in Excel called "product" with syntax Product(x:y), but it really needs indirect references to be used to work with such formulae. I will need to figure that one out.

- On the other hand, the recursive formulae are dead easy to work with in MS Excel. I could work up a demonstration using those.
- It would be interesting to compare these formulae with the formulae for DCFs.
- Would I get the same formulae if I do the analysis on growth from the bottom up using η in place of $\mathcal{R}$? I should get the same answers.

5.1 - Some Doubts and Questions

- Recursive formulae that involve three variables (like I, C and B) require at least one bridging formula for each step (accessing previous values) or some external input (such as δ_i or R_i or η_i). I have implemented both kinds (bridge and external guidance), giving the appearance of recursion but having significant external 'guidance' of the developing process from the sequence of efficiencies. It is not a true determinate recursion such as the Fibonacci sequence. In fact, there are unlimited degrees of freedom in my formulae and they can be used to emulate any curve, no matter how noisy. So, there is not a lot of real meaning in them unless you restrict that stream of external inputs in some way, such as defining δ as a function of time t. The trick that would make the formulae more useful would be to find such a definition that makes the recursion more determinate. Is there such a function, or set of functions (d(t), h(t) or R(t)) that make the complicated formulae both more simple and more useful.
- That being said, within the recursive formulae there is an implied order in which the three variables I, C and B are calculated, and that order is somewhat arbitrary, altering the sequence of values slightly in each case. There are six ways to order three variables (6=3x2x1) so the formulae developed in this note are possibly just one set of six possible sets. Is there one of the six sets of potential formulae that is more useful, or more easy to work with?
- It seems to me that recursion in time exists in two forms: repeated capture/degradation/dispersal by one organism, and repeated decay as energy passes through many organisms. I have addressed both types in this note, but not clearly distinguishing between them. The second is also analogous to a flow of energy from well-head to point of use. The cumulative formulae from start to iteration n would be point-of-use formulae.